On the power of Statistical Zero Knowledge

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Most graphics are credited to Adam Bouland

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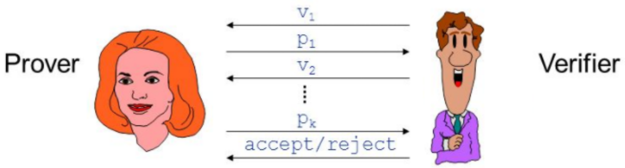
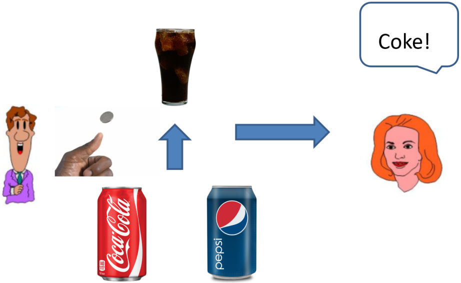
Zero Knowledge Proof **[Goldwasser Micali Rackoff '84]**



- Alice wants to convince Bob that a certain statement is true,
  - but doesn't want him to know anything more.

Example : Coke-Pepsi Challenge



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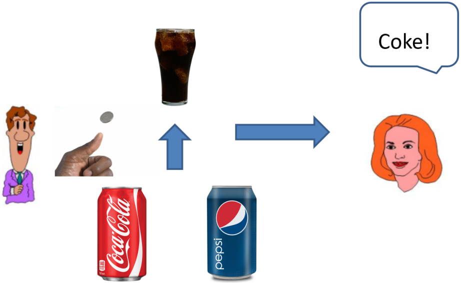

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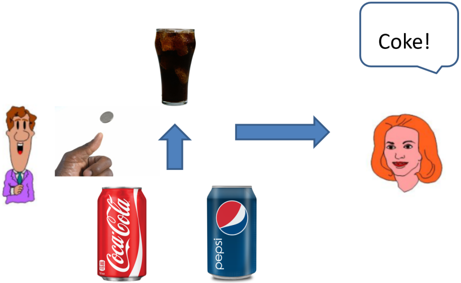

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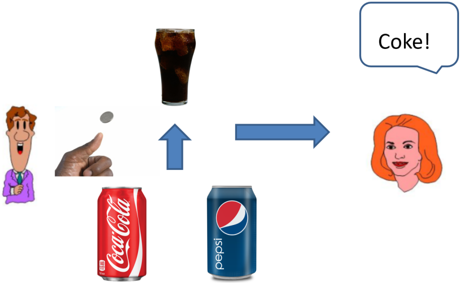

- Alice wants to convince Bob that coke and pepsi are different.
- **Protocol**: Bob flips a random coin, secretly pours coke or pepsi into a glass.
- Alice answers whether it is coke or pepsi.
- **Zero knowledge**: since Bob already knew the answer.

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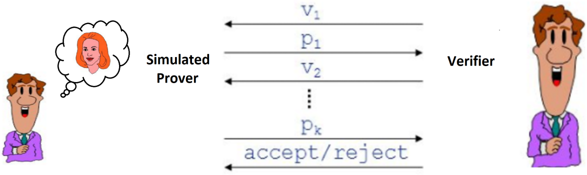
- All information Bob gets from Alice is a (distribution of) conversation which convinced him.
- $\Pi_{A \leftrightarrow B}$: the distribution of the conversation between Alice and Bob.
- $\Leftrightarrow$ Bob can produce a distribution of the conversation Π_B which "looks like" $\Pi_{A \leftrightarrow B}$. (In the YES case.)



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- Indeed, our results apply for the following sub-class of SZK.
- (**Non-Interactive Statistical Zero Knowledge Proof**) NISZK : Alice doesn't interact with Bob, just say something and leave (they share public random bits)

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- Evidence that SZK contains some very hard problems.
- Relationship between several different kinds of proof systems related to SZK.

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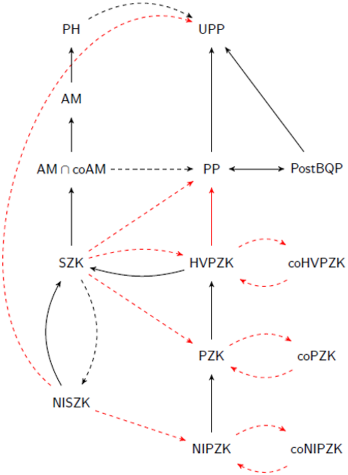
- And more!

New Oracle Separations (Result I & III)



solid line : containment

dashed line : separation

black : known results

red : new results

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  - Quadratic Residuosity.
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- What is the evidence that SZK contains some really hard problems?
- **Obstacle:** $\mathsf{P} \neq \mathsf{SZK}$ implies $\mathsf{P} \neq \mathsf{NP}$
  - $\mathsf{P} = \mathsf{NP} \implies \mathsf{P} = \mathsf{PH}$ and $\mathsf{SZK} \subseteq \mathsf{PH}$.

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- **[Watrous'02]**: Does relativized SZK contain problems outside of PP? (PP is the smallest natural classical class containing BQP.)

Probabilistic Polynomial-Time (PP)

- Languages decidable by poly-time randomized algorithms with unbounded error.
  - If Yes: $\Pr[\text{accept}] > 1/2$.
  - If No: $\Pr[\text{accept}] < 1/2$.
  - Gap may be exponentially small. (because there is only polynomial number of coin flips).
- PP is very powerful : PP contains NP and P^{PP} contains PH by **[Toda'91]**.

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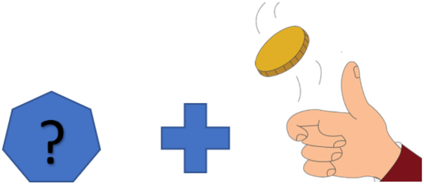

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- UPP query complexity is equivalent to
    - **Threshold Degree of f**: $\deg_{\pm}(f)$, the least degree polynomial p which sign-represents f
    - $p(x) > 0$ when $f(x) = 1$, and $p(x) < 0$ when $f(x) = 0$.

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- A brief overview of how is it proved.

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- **Sad reality**: PP is too powerful.

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- **Hard for** PP: PP is not closed under randomized reduction for some oracle $\mathcal{O}$.
  - In fact, $(\mathsf{BP} \cdot \mathsf{NP})^{\mathcal{O}} = \mathsf{AM}^{\mathcal{O}} \not\subset \mathsf{PP}^{\mathcal{O}}$ **[Vereshchagin'92]**.

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- **Captures what can be randomized reduced to f.**
- **Intuition**:
  - Since randomized reduction is hard for PP, $\mathrm{GapMaj}_d(f)$ should be harder than f for PP in some sense.

Core Technique Result: Hardness Amplification Theorem Gapped Majority is really hard for PP

f : requires a degree d poly to approximate within L_∞ distance 0.1

|f(x) – p(x)| <= 0.1

→

F(GapMaj(f)) : requires a degree $\Omega(d)$ poly to sign-represents it.

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- Proved by constructing the dual object to witness the high threshold degree. (**[Sherstov'14]**,**[Bun and Thaler'15]**).

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F(GapMaj(f)) : requires a degree $\Omega(d)$ poly to sign-represents it.

p(x) > 0 if f(x) = 1
p(x) < 0 if f(x) = 0

Composition with Gapped-Majority:
GapMaj(f):
d copies of f on inputs
x_1,x_2,…,x_d
1 when 2/3 of f(x_i)'s are 1
0 when 2/3 of f(x_i)'s are 0

- Proved by constructing the dual object to witness the high threshold degree. (**[Sherstov'14]**,**[Bun and Thaler'15]**).
- Actually it has a converse, when f has a degree d L_∞-approximate-polynomial, $\mathrm{GapMaj}_d(f)$ has threshold degree $O(d)$.

SZK$^{\mathcal{O}} \not\subseteq$ UPP$^{\mathcal{O}}$

- Collision : Distinguish whether a given function from $[n]$ to $[n]$ is 1-to-1 or 2-to-1.
    - constant query SZK protocol.
    - require $\Omega(n^{1/3})$ (bounded) approximate polynomial degree. **[Aaronson'02],[Aaronson and Shi'04],[Ambainis'05],[Kutin'05]**

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- Implies our separation.

Result II : Communication SZK is very powerful

- **Result 2**: SZK^{cc} (even NISZK^{cc}) is not contained in UPP^{cc}.

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- **Result 2**: $\mathsf{SZK}^{\mathrm{cc}}$ (even $\mathsf{NISZK}^{\mathrm{cc}}$) is not contained in $\mathsf{UPP}^{\mathrm{cc}}$.
- Answers **[Göös, Pitassi and Watson'15]**.
  - **[GPW'15]** : can we show $(\mathsf{AM}^{\mathrm{cc}} \cap \mathsf{coAM}^{\mathrm{cc}}) \not\subseteq \mathsf{UPP}^{\mathrm{cc}}$?
  - $\mathsf{SZK} \subseteq (\mathsf{AM}^{\mathrm{cc}} \cap \mathsf{coAM}^{\mathrm{cc}}) \subseteq \mathsf{AM}^{\mathrm{cc}}$.

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- $\mathrm{AM}^{\mathrm{cc}}$: Notoriously hard to prove a communication complexity lower bound against it (first step toward proving lower bound for $\mathrm{PH}^{\mathrm{cc}}$).
- $\mathrm{UPP}^{\mathrm{cc}}$: the **strongest class** we know how to prove non-trivial communication lower bound.

Result II : Communication SZK is very powerful

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$\not\supseteq$

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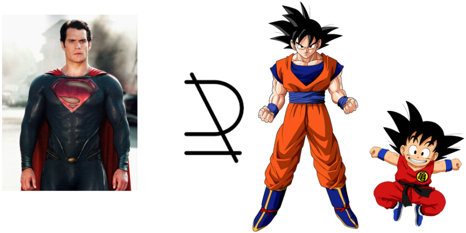

$\mathrm{UPP}^{\mathrm{CC}}$ $\not\supseteq$ $\mathrm{AM}^{\mathrm{CC}}$ $\mathrm{NISZK}^{\mathrm{CC}}$

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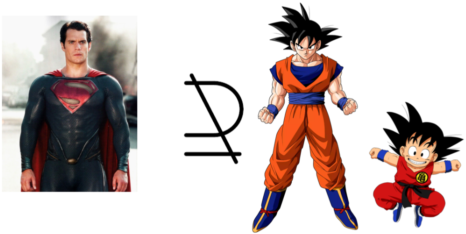

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    - **[Klauck'2011]**: (AM^{cc} $\cap$ coAM^{cc}) $\not\subseteq$ PP^{cc}.

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  - **[Klauck'2011]**: $(\text{AM}^{\text{cc}} \cap \text{coAM}^{\text{cc}}) \not\subseteq \text{PP}^{\text{cc}}$.
  - **Our improvement** : $\text{NISZK}^{\text{cc}} \not\subseteq \text{UPP}^{\text{cc}}$, $\text{NISZK}^{\text{cc}} \subseteq \text{SZK}^{\text{cc}} \subseteq \text{AM}^{\text{cc}}$.

Result II : Communication SZK is very powerful

- **Moral** : Communication SZK contains some very hard problems(even outside of UPP), which explains why we can't prove lower bounds for $\mathrm{AM}^{\mathrm{cc}}$.

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SZK and its friends

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- **Perfect** Zero Knowledge (PZK) : the simulated prover looks exactly the same as Alice.
- **Non-Interactive** Zero Knowledge (NISZK or NIPZK) : no interaction, Alice says something and just leave. (they share some public random bits).

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What is the relationship between these classes?

- Two intriguing open questions here:
  - Is SZK equal to PZK (or at least an **oracle separation**)? **[Aiello Hastad'91]**
  - Is PZK closed under complement, the same way that SZK is **[Sahai Vadhan'99]** (or at least an **oracle separation**)?

Our Result

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  - $\mathsf{coPZK}^{\mathcal{O}} \neq \mathsf{PZK}^{\mathcal{O}}$.
  - $\mathsf{coNIPZK}^{\mathcal{O}} \neq \mathsf{NIPZK}^{\mathcal{O}}$.
- Therefore SZK may be more powerful than PZK, and any proof that SZK = PZK, or PZK = coPZK, must be nonrelativizing.

Technique

- **Lemma**: $\mathsf{PZK}^{\mathcal{O}} \subseteq \mathsf{PP}^{\mathcal{O}}$, relative to all oracle $\mathcal{O}$.

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Technique

- **Lemma**: $\mathsf{PZK}^{\mathcal{O}} \subseteq \mathsf{PP}^{\mathcal{O}}$, relative to all oracle $\mathcal{O}$.
  - $\mathsf{SZK}^{\mathcal{O}} \not\subseteq \mathsf{PP}^{\mathcal{O}} \implies \mathsf{SZK}^{\mathcal{O}} \neq \mathsf{PZK}^{\mathcal{O}}$.
- For $\mathsf{PZK}^{\mathcal{O}} \neq \mathsf{coPZK}^{\mathcal{O}}$, we use a different proof with another hardness amplification theorem.

Thanks!