# CS 278: Computational Complexity Theory
# Lecture Notes - September 4, 2025

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**Definition 1** (Non-deterministic time with bounded guess)**.** *For a function $T, G : \mathbb{N} \to \mathbb{N}$, we define NTIMEGUESS$[T(n), G(n)]$ to be the class of languages that can be decided by a non-deterministic multi-tape Turing machine in time $O(T(n))$ with making at most $G(n)$ non-deterministic guesses.*

- *That is, $L \in$ NTIMEGUESS$[T(n), G(n)]$ if there exists a non-deterministic multi-tape Turing machine $M$ and a constant $c$ such that:*
  - *$M$ decides $L$ (i.e., $M$ accepts $x$ if and only if $x \in L$)*
  - *For all inputs $x$ of length $n$, $M$ halts within $c \cdot T(n)$ steps and makes at most $G(n)$ non-deterministic guesses.*

**Theorem 2** (Non-deterministic time hierarchy theorem with bounded guess)**.** *Let $T, G, W : \mathbb{N} \to \mathbb{N}$ be time-constructible functions such that $G(n) = o(T(n))$ and $W(n) = o(n)$. Then there is a language $L \in$ NTIME$[T(n)]$ but $L$ is almost-everywhere separated from NTIMEGUESS$[G(n), W(n)]$.*

*Proof.* Let $M$ be a machine from NTIMEGUESS$[G(n), W(n)]$. That is, there is a deterministic machine $D_M$ that takes $x \in \{0,1\}^n$ and $w \in \{0,1\}^{W(n)}$, runs in $O(G(n))$ time such that $M(x) = 1$ if and only if $D_M(x, w) = 1$ for some $w \in \{0,1\}^{W(n)}$. Here we call $w$ the witness.

For $n \geq 10 \cdot |\langle M \rangle|$ and $w \in \{0,1\}^{n/10}$, we define $\langle M, w \rangle_n$ to be an encoding of the pair $\langle M \rangle$ (the description of $M$) and $w$ (the witness), such that $\langle M, w \rangle_n \in \{0,1\}^n$.

For a $M \in$ NTIMEGUESS$[G(n), W(n)]$, we will construct a hard language $H \in$ NTIME$[T(n)]$ such that for every $n \geq 10 \cdot |\langle M \rangle|$, it holds that $M$ does not compute $H$ on inputs of length $n$.

Let $w_1, \ldots, w_{2^{n/10}}$ be a sequence of all possible strings of length $n/10$. For simplicity, we assume $D_M$ takes exactly $n/10$ bits as witness.

We define $H$ as follows:

$$H(x) = \begin{cases} M(\langle M, w_{i+1} \rangle_n) \wedge [D_M(\langle M, w_1 \rangle_n, w_i) = 0] & \text{if } x = \langle M, w_i \rangle_n \text{ for some } i \in [1, 2^{n/10}) \\ [D_M(\langle M, w_1 \rangle_n, w_{2^{n/10}}) = 0] & \text{if } x = \langle M, w_{2^{n/10}} \rangle_n \\ 0 & \text{otherwise} \end{cases}$$

Here, $[D_M(\langle M, w_1 \rangle_n, w_i) = 0]$ means we treat $D_M(\langle M, w_1 \rangle_n, w_i) = 0$ as a proposition, $[D_M(\langle M, w_1 \rangle_n, w_i) = 0]$ is true if "$D_M(\langle M, w_1 \rangle_n, w_i) = 0$" holds.

Note that $H \in \text{NTIME}[T(n)]$ since $G(n) = o(T(n))$.

Now, for the sake of contradiction, suppose $M$ computes $H$ on all $n$-bit inputs. Then for every $i \in [1, 2^{n/10})$, we have

$$H(\langle M, w_i \rangle_n) = H(\langle M, w_{i+1} \rangle_n) \wedge [D_M(\langle M, w_1 \rangle_n, w_i) = 0]$$

and also

$$H(\langle M, w_{2^{n/10}} \rangle_n) = [D_M(\langle M, w_1 \rangle_n, w_{2^{n/10}}) = 0]$$

Chaining the above together, we have

$$H(\langle M, w_1 \rangle_n) = \bigwedge_{i=1}^{2^{n/10}} [D_M(\langle M, w_1 \rangle_n, w_i) = 0]$$

Reading the right side above, it is true if and only if $D_M(\langle M, w_1 \rangle_n, w_i) = 0$ for all $i \in [1, 2^{n/10}]$, which means $M$ rejects the input $\langle M, w_1 \rangle_n$.

Therefore, we have $H(\langle M, w_1 \rangle_n) = \neg M(\langle M, w_1 \rangle_n)$, which contradicts the assumption that $M$ computes $H$ on all $n$-bit inputs.

$\square$