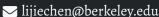
Computational Complexity Theory Fall 2025

Time complexity and Hierarchy theorems September 2 and 4, 2025

Lijie Chen

University of California, Berkeley



• Office Hours: 2:00 - 3:00 PM, SODA 627, Tuesday

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- Additional Office Hours?: 11:30 AM 12:30 PM, SODA 627, Thursday

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- Course website: https://chen-lijie.github.io/cs278-complexity.html

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- Computation ends when machine reaches q_{accept} or q_{reject}

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 - You can **sort** and evaluate a **circuit** in time O(n log n) on multi-tape TM, so they are indeed quite powerful!
 - o Later in this course, we will study Ryan Williams' breakthrough results on T-time in \sqrt{T} -space, which holds for multi-tape TM.

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- d-dimensional multi-/single-tape Turing machines have d-dimensional memory, and can move the tape heads to any neighboring cell in one step.
- Pointer machines allow the machine to maintain "pointers" to arbitrary cells in the memory (instead of directly accessing the memory as the RAMs).

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- The proof is technical (and won't be really needed for this course), can be found in Chapter 1 of Arora-Barak.

Theorem (Universal multi-tape TM with time bound T)

There exists a universal multi-tape Turing machine U_{clock} such that for any multi-tape Turing machine M, input x, and time bound T:

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• The same is true for other variants of Turing machines.

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Definition (DTIME)

Time hierarchy theorem for deterministic time

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Remark

• In many situations, you want $T : \mathbb{N} \to \mathbb{N}$ to be time-constructible, i.e., there exists a deterministic multi-tape Turing machine that can compute T(n) given (binary encoded input) n in time O(T(n)).

Theorem

• Let $T : \mathbb{N} \to \mathbb{N}$ be a time-constructible function. There is a language $L \in DTIME[T(n) \log^2 T(n)]$ but $L \notin DTIME[T(n)]$.

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- Essentially the same proof for the hardness of the halting problem.

Definition (Multi-tape non-deterministic Turing machine)

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- M accepts x if and only if there is a sequence of transitions that leads to q_{acc}.

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- **Question**: Can you think about why we cannot use the same proof as the deterministic time hierarchy theorem?
- It seems for deterministic time hierarchy theorem, we only used the existence of universal Turing machine? And such machine exists for non-deterministic time as well?
- **Answer**: "Let $H(M) := \neg U_{\text{clock}}(\langle M, M, \widetilde{T}(n) \rangle)$." This \neg cannot be done for non-deterministic Turing machine!

Theorem (Non-deterministic Time Hierarchy Theorem)

• Let $f,g: \mathbb{N} \to \mathbb{N}$ be time-constructible functions such that f(n+1) = o(g(n)). Then

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• **Proof:** see the white board!

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- **Big open question**: prove an almost everywhere separation between $NTIME[n^2]$ and $NTIME[2^n]$.

Definition (Non-deterministic time with bounded guess)

• For a function $T, G : \mathbb{N} \to \mathbb{N}$, we define NTIMEGUESS[T(n), G(n)] to be the class of languages that can be decided by a non-deterministic multi-tape Turing machine in time O(T(n)) with making at most G(n) non-deterministic guesses.

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 - For all inputs x of length n, M halts within $c \cdot T(n)$ steps and makes at most G(n) non-deterministic guesses.

Theorem (Non-deterministic time hierarchy theorem with bounded guess)

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• **Proof:** see the white board!